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THE MINIMAL N=2 SUPEREXTENSION OF THE NLS EQUATION

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Abstract. We show that the well known $N = 1$ NLS equation possesses $N = 2$ supersymmetry and thus it is actually the $N = 2$ NLS equation. This supersymmetry is hidden in terms of the commonly used $N = 1$ superfields but it becomes manifest after passing to the $N = 2$ ones. In terms of the new defined variables the second Hamiltonian structure of the supersymmetric NLS equation coincides with the $N = 2$ superconformal algebra and the $N = 2$ NLS equation belongs to the $N = 2$ $a = 4$ KdV hierarchy. We propose the KP-like Lax operator in terms of the $N = 2$ superfields which reproduces all the conserved currents for the corresponding hierarchy.

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Introduction. Recently, it has been realized [1-4] that many integrable two dimensional equations can be supersymmetrized by considering the supersymmetric extensions of their second Hamiltonian structures. In particular, the $N=2$ supersymmetric Boussinesq equation [2] and $N=3,4$ KdV equations [3,4] have been constructed along this line starting from the classical $N=2$ W_3 algebra and $N=3,4$ superconformal algebras, respectively.

In contrast, the known $N=2$ supersymmetric extensions of the nonlinear Schrödinger equation (NLS) [5,6] have been constructed in a different way starting from some *ad hoc* assumptions about superfield content of the theory. The key problem in the $N=2$ supersymmetrization of the NLS equation [5,6] is the lack of the $N=2$ superextension of its second Hamiltonian structure which is connected in the bosonic case with the reduced $sl(2)$ Kac-Moody algebra [5].

The idea of our construction of the $N=2$ supersymmetric extension of the NLS equation comes from a disguised form of the bosonic NLS equation [7,8] whose second Hamiltonian structure is the $U(1)$ Kac-Moody extension of the Virasoro algebra. It is more or less evident that the simplest $N=2$ supersymmetric extension of this second Hamiltonian structure is the $N=2$ superconformal algebra.

In this letter, we will explore this idea by explicit constructing of the minimal $N=2$ supersymmetric NLS equation containing two bosonic and two fermionic fields (i.e. twice as small as in [5,6]). Moreover, as we will show later, this equation can be connected with the $N=1$ NLS equation [9,10] through the Miura-like transformation. We propose the Lax operator in terms of the $N=2$ superfield and discuss some possible generalizations to higher supersymmetry. We also demonstrate that the $N=1$ NLS equation possesses the hidden global $N=2$ supersymmetry and it can be rewritten in terms of the $N=2$ chiral–anti-chiral superfields.

Minimal N=2 super NLS equation. Let us start with some brief recalling of the salient features of the description of the NLS equation within the multi-field representation of the KP hierarchy [7,8].

The NLS hierarchy can be defined via the Lax operator [7,8]

$$L=\partial+R\frac{1}{\partial-S} \tag{1}$$

and the flows

$$\frac{\partial L}{\partial t}=\left[L_{+}^{n},L\right] \ , \tag{2}$$

where $\{+\}$ denotes the purely differential part of the nth power of the Lax operator. Starting from this Lax operator we can define the first Hamiltonian structure[1]

$$R(z_1)S(z_2)=\frac{1}{z_{12}^2}, \tag{3}$$

with the Hamiltonian H_3

$$H_3=\int dx(R^2+RS^2+R'S), \tag{4}$$

[1]We use the OPE technique instead of the Poisson brackets.

and the second ones

$$
\begin{aligned}
R(z_1)R(z_2) &= \frac{2R(z_2)}{z_{12}} + \frac{R'(z_2)}{z_{12}^2} \, , \\
S(z_1)S(z_2) &= -\frac{2}{z_{12}^2} \, , \\
R(z_1)S(z_2) &= \frac{2}{z_{12}^3} + \frac{S(z_2)}{z_{12}} + \frac{S'(z_2)}{z_{12}^2}
\end{aligned} \tag{5}
$$

with Hamiltonian H_2

$$
H_2 = \int dx RS. \tag{6}
$$

for the first nontrivial flow equations [7,8]

$$
\frac{\partial S}{\partial t} = -S'' + 2S'S + 2R' \quad , \quad \frac{\partial R}{\partial t} = R'' + 2(RS)' \quad , \tag{7}
$$

which are a disguised form of the NLS equation. Namely, the standard form of the NLS equation

$$
\frac{\partial u}{\partial t} = u'' - u^2 v \quad , \quad \frac{\partial v}{\partial t} = -v'' + v^2 u \quad , \tag{8}
$$

coincides with (7) after passing to the new fields S and R defined as

$$
R = -\frac{1}{2}uv \quad , \quad S = -\frac{v'}{v} \, . \tag{9}
$$

Let us note that the standard complex conjugated rules for u, v and t

$$
u^* = v \ , \ v^* = u \ , \ t^* = -t \tag{10}
$$

have the following representation in terms of R and S:

$$
R^* = R \ , \ S^* = -S - \frac{R'}{R} \, . \tag{11}
$$

Quite unexpectedly, the standard conjugated rules for R and S

$$
R^* = R - S' \ , \ S^* = -S \ , \ t^* = -t \tag{12}
$$

induce the Bäcklund-Schlesinger transformations for the u, v system [11]

$$
u^* = v\left(uv - 2(\log v)''\right) \ , \ v^* = \frac{1}{v} \, , \tag{13}
$$

which are useful for solving the NLS equation.

It is easy to recognize that the second Hamiltonian structure (5) for the NLS equation in the form (7) is an extension of the Virasoro algebra with the stress-tensor R by the $U(1)$ current S in the non-primary basis.

This Hamiltonian structure can be immediately extended to the $N=2$ supersymmetric case. Indeed, the $N=2$ bosonic supercurrent J generating the $N=2$ superconformal algebra (SCA) through the following SOPE's

$$
J(Z_1)J(Z_2) = \frac{c/4}{Z_{12}^2} + \frac{\bar\theta_{12}\overline{\mathcal{D}}J(Z_2)}{Z_{12}} - \frac{\theta_{12}\mathcal{D}J(Z_2)}{Z_{12}} + \frac{\theta_{12}\bar\theta_{12}J(Z_2)}{Z_{12}^2} + \frac{\theta_{12}\bar\theta_{12}J(Z_2)'}{Z_{12}} \, , \tag{14}
$$

where

$$Z = (z, \theta, \bar{\theta}),\ \theta_{12} = \theta_1 - \theta_2,\ \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2,\ Z_{12} = z_1 - z_2 + \frac{1}{2}\left(\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1\right) \tag{15}$$

and $\mathcal{D}, \overline{\mathcal{D}}$ are the spinor covariant derivatives

$$\mathcal{D} = \frac{\partial}{\partial\theta} - \frac{1}{2}\bar{\theta}\frac{\partial}{\partial z} \quad , \quad \overline{\mathcal{D}} = \frac{\partial}{\partial\bar{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z} \tag{16}$$

$$\left\{\mathcal{D}, \overline{\mathcal{D}}\right\} = -\frac{\partial}{\partial z} \quad , \quad \{\mathcal{D}, \mathcal{D}\} = \left\{\overline{\mathcal{D}}, \overline{\mathcal{D}}\right\} = 0,$$

contains in the bosonic sector (after putting all fermions equal to zero) just the $U(1)$ extension of the Virasoro algebra, i.e. the second Hamiltonian structure of the $N = 0$ NLS equation (5) (in the primary basis). Therefore, it is natural to suppose that the second Hamiltonian structure of the $N = 2$ superextension of the NLS equation we are looking for coincides with $N = 2$ SCA (14). Due to the dimensionality of the Hamiltonian H_2 (6) in the bosonic case (cm^{-2}), there is a unique candidate to be its $N = 2$ superextension:

$$sH_2 = \int dz d\theta d\bar{\theta} J(Z) J(Z) \quad , \tag{17}$$

and the equation of motion is easy to compute

$$\frac{\partial J}{\partial t} = -\frac{c}{4}\left[\mathcal{D}, \overline{\mathcal{D}}\right] J' + 4J'J. \tag{18}$$

It will be of importance that the central charge c is nonzero in (18), but its concrete value is nonessential and can be changed by rescaling of J and t. Henceforth, we fix the central charge equal to $c = -4$ and call equation (18) the minimal $N = 2$ supersymmetric NLS equation.

Before going further and to avoid any failure to understand, let us clarify some points.

First of all eq. (18) is not a new one because it belongs to the integrable $N = 2$ KdV hierarchy with the parameter $a = 4$ [12]. So, the main question we have to answer immediately is why we call this equation the $N = 2$ supersymmetric extension of the NLS one? To answer, let us pass from the superfield equation (18) to the component ones

$$\begin{aligned}
\frac{\partial S}{\partial t} &= -S'' + 2S'S + 2R' \quad , \\
\frac{\partial R}{\partial t} &= R'' + 2(RS)' + 8(\xi\bar{\xi})' \quad , \\
\frac{\partial \xi}{\partial t} &= \xi'' + 2(\xi S)' \quad , \\
\frac{\partial \bar{\xi}}{\partial t} &= -\bar{\xi}'' + 2(\bar{\xi} S)',
\end{aligned} \tag{19}$$

where the component currents are defined as follows:

$$S = 2J| \ , \ R = \left(\left[\mathcal{D}, \overline{\mathcal{D}}\right] J + J'\right)| \ , \ \xi = \mathcal{D}J| \ , \ \bar{\xi} = \overline{\mathcal{D}}J| \tag{20}$$

and the sign $|$ means putting $\theta, \bar{\theta}$ equal to zero.

Now it is the matter of calculations to see that equations (19) are of the disguised form of the $N=1$ NLS equation [9,10]. Indeed, the $N=1$ NLS equation has the following form:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= u'' - u^2 v - u(\psi\bar{\psi}' - \psi'\bar{\psi}) - u'\psi\bar{\psi} \quad , \\
\frac{\partial v}{\partial t} &= -v'' + uv^2 + v(\psi\bar{\psi}' - \psi'\bar{\psi}) - v'\psi\bar{\psi} \quad , \\
\frac{\partial \psi}{\partial t} &= \psi'' - \psi uv + \psi\psi'\bar{\psi} \quad , \\
\frac{\partial \bar{\psi}}{\partial t} &= -\bar{\psi}'' + \bar{\psi}uv - \psi\bar{\psi}\bar{\psi}'.
\end{aligned} \tag{21}$$

After passing to the new fields defined as

$$\begin{aligned}
S &= -\frac{1}{2}\psi\bar{\psi} - \frac{v'}{v}, \\
R &= -\frac{1}{2}\psi\bar{\psi}' - \frac{1}{2}uv, \\
\xi &= \frac{1}{4}\psi v, \\
\bar{\xi} &= -\frac{1}{4}\bar{\psi}u + \frac{\bar{\psi}''}{2v} - \frac{\bar{\psi}'v'}{2v^2}
\end{aligned} \tag{22}$$

we recover equations (19).

Thus, we proved that our $N=2$ NLS equation is nothing else but the disguised form of the $N=1$ NLS one and possesses the manifest $N=2$ superconformal symmetry. Let us stress that the transformations (22) are Miura-like ones. So, equations (19) are equivalent to the $N=1$ NLS equations (21) in the same sense as the KdV equation is equivalent to the mKdV one.

Now, recognizing the second Hamiltonian structure of the $N=2$ NLS equation as $N=2$ SCA it is natural to generalize it for higher supersymmetries, considering the $N=3,4$ SCA's. The corresponding $N=3,4$ superextensions of the NLS equation will be the first non trivial equations in the $N=3,4$ super KdV hierarchies [3,4]. The detailed calculations for these cases would be presented elsewhere.

Let us finish this letter with two comments.

First, it can be verified that the Lax operator for our $N=2$ NLS equation has the following interesting form:

$$L = \hat{L} + L^{\frac{1}{2}} \quad , \tag{23}$$

where $\hat{L}$ and $L^{\frac{1}{2}}$ are two different square roots from the Lax operator L_{KdV} for the $a=4$ $N=2$ super KdV equation [12]

$$\begin{aligned}
L_{\mathrm{KdV}} &= \partial^2 + 2J\left[\mathcal{D},\overline{\mathcal{D}}\right] + 2(\mathcal{D}J)\overline{\mathcal{D}} - 2(\overline{\mathcal{D}}J)\mathcal{D} + \left(\left[\mathcal{D},\overline{\mathcal{D}}\right]J\right) + J^2, \\
\hat{L} &= \left[\mathcal{D},\overline{\mathcal{D}}\right] + J
\end{aligned} \tag{24}$$

and $L^{\frac{1}{2}}$ is a pseudodifferential operator starting with ∂:

$$L^{\frac{1}{2}} = \partial + \left(\frac{1}{2}\left(\left[\mathcal{D},\overline{\mathcal{D}}\right]J\right) + J\left[\mathcal{D},\overline{\mathcal{D}}\right] + (\mathcal{D}J)\overline{\mathcal{D}} - (\overline{\mathcal{D}}J)\mathcal{D}\right)\partial^{-1} + \ldots \quad . \tag{25}$$

The corresponding Lax equation has the standard form:

$$\frac{\partial L}{\partial t} = \left[L_+^2, L \right] . \tag{26}$$

It is interesting enough that using our Lax operator (23) all the $N = 2$ super KdV hierarchy with the parameter $a = 4$ can be obtained from the following equations:

$$\frac{\partial L}{\partial t} = \left[L_+^n, L \right] \tag{27}$$

in contrast with [12] where only the equations with even n are reproduced. Our Lax operator (23) is a natural generalization of the Lax pairs representation proposed in [13] for each of the $N = 2$ super KdV hierarchy equations and describes some reduction of the $N = 2$ KP hierarchy.

Finally, we would like to stress that the $N = 1$ NLS equation (21) can be rewritten in terms of the $N = 2$ chiral-anti-chiral fermionic superfields $F(Z), \bar{F}(Z)$ as

$$\begin{aligned} \frac{\partial F}{\partial t} &= F'' - F\mathcal{D}(\bar{F}\overline{\mathcal{D}}F), \\ \frac{\partial \bar{F}}{\partial t} &= -\bar{F}'' + \bar{F}\overline{\mathcal{D}}(F\mathcal{D}\bar{F}), \\ \mathcal{D}F &= \overline{\mathcal{D}}\bar{F} = 0, \ F^* = \bar{F} . \end{aligned} \tag{28}$$

where the components of superfields F and $\bar{F}$ are defined as follows:

$$u = \overline{\mathcal{D}}F| \ , \ v = \mathcal{D}\bar{F}| \ , \ \psi = F| \ , \ \bar{\psi} = \bar{F}| \quad . \tag{29}$$

In terms of the superfields F and $\bar{F}$ the transformations (22) have the following form:

$$J = -\frac{1}{4}F\bar{F} - \frac{1}{2}\frac{\left(\mathcal{D}\bar{F}\right)'}{\mathcal{D}\bar{F}} . \tag{30}$$

This demonstrates the hidden $N = 2$ supersymmetry of the $N = 1$ NLS equation (21), but in this form it is unclear how to promote this global $N = 2$ supersymmetry to the $N = 2$ superconformal one which is manifest in terms of the $N = 2$ superfield $J(Z)$ (18).

Conclusion. In this paper, we have shown that the well known $N = 1$ NLS equation possesses $N = 2$ supersymmetry and thus it is actually the $N = 2$ NLS equation. This supersymmetry is hidden in terms of the commonly used $N = 1$ superfields but it becomes manifest after passing to the $N = 2$ ones. In terms of the new defined variables the second Hamiltonian structure of the supersymmetric NLS equation coincides with the $N = 2$ superconformal algebra and the $N = 2$ NLS equation belongs to the $N = 2$ $a = 4$ KdV hierarchy. We also constructed the KP-like Lax operator in terms of the $N = 2$ superfields which reproduces all the conserved currents for the corresponding hierarchy. The relation between the $N = 2$ NLS and $N = 2$ KdV equations can be easily promoted to higher supersymmetries. We postpone the discussion of the NLS equations with higher supersymmetries to future publications.

In the forthcoming paper [14] we will show that the appearance of the hidden $N=2$ supersymmetry in the $N=1$ NLS equation has a nice geometric description in the framework of the coset approach.

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